Resurgence and Large N

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CAQCD, Minnesota, May 2016

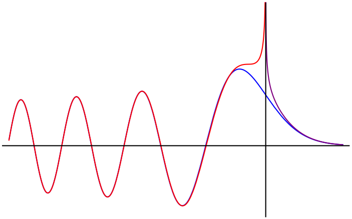
GD, Gökçe Başar, Mithat Ünsal, 1306.4405, 1501.05671, 1603.04924

P. Buividovich, GD, S. Valgushev, 1512.09021

Physical Motivation

- how to define (& calculate!) a Minkowski path integral?

$$\int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \quad \text{versus} \quad \int \mathcal{D}A \, \exp\left(-\frac{1}{\hbar} S[A]\right)$$



$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}t^3 + x\,t\right)}\,dt \sim \begin{cases} \dfrac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\,x^{1/4}} \,, & x \to +\infty \\[2em] \dfrac{\sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-x)^{1/4}} \,, & x \to -\infty \end{cases}$$

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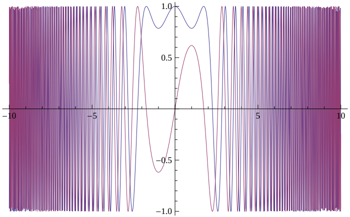

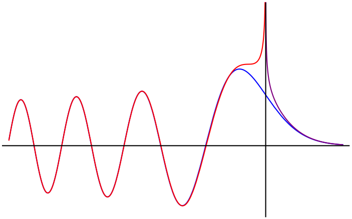

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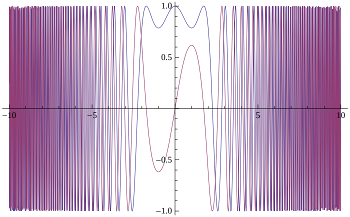
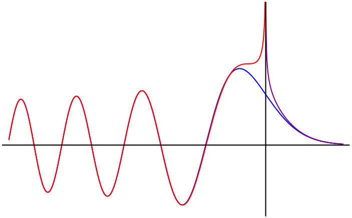




$$\int \mathcal{D}A \, e^{-\frac{1}{g^2} S[A]} = \sum_{\text{thimbles}\, k} \mathcal{N}_k \, e^{-\frac{i}{g^2}\, S_{\rm imag}[A_k]} \int_{\Gamma_k} \mathcal{D}A \, e^{-\frac{1}{g^2} S_{\rm real}[A]}$$

Generic Resurgent Relations within Trans-series

$$
\begin{aligned}
F(g^2) \quad \sim \quad & \left(f_0^{(0)} + f_1^{(0)} g^2 + f_2^{(0)} g^4 + \dots\right) \\
& + \sigma\, e^{-S/g^2} \left(f_0^{(1)} + f_1^{(1)} g^2 + f_2^{(1)} g^4 + \dots\right) \\
& + \sigma^2\, e^{-2S/g^2} \left(f_0^{(2)} + f_1^{(2)} g^2 + f_2^{(2)} g^4 + \dots\right) + \dots
\end{aligned}
$$

- basic idea: non-perturbative contributions from Borel summation of divergent series in one sector are related to terms in other non-perturbative sectors

- generic for ODEs, difference equations, dynamical systems, (some) PDEs, exponential integrals, matrix integrals, ...

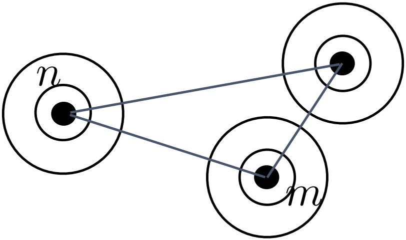
- QM and matrix models

- asymptotically free QFT: 2d σ-models: $\mathbb{CP}^{N-1}$, $Gr(M,N)$, PCM, $O(N)$, dPCM

- SUSY/Localizable QFT

resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities

J. Écalle, 1980



Resurgence from Exponential Integrals (Berry/Howls 1991, Howls 1992)

- all-orders steepest descents for contour integrals:

$$I^{(n)}(g^2) = \int_{C_n} dz\, e^{-\frac{1}{g^2} f(z)} = \frac{1}{\sqrt{1/g^2}}\, e^{-\frac{1}{g^2} f_n}\, T^{(n)}(g^2)$$

- $T^{(n)}(g^2)$: beyond the usual Gaussian approximation

- asymptotic expansion of fluctuations about the saddle n:

$$T^{(n)}(g^2) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, g^{2r}$$

- exact resurgent relation between fluctuations about n^{th} saddle and about neighboring saddles m; here $F_{nm} \equiv f_m - f_n$

$$T_r^{(n)} = \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(-1)^{\gamma_{nm}}}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \dots \right.$$

Resurgence from Exponential Integrals: example

$$I(g^2) = \int_0^{\pi} dz\, e^{-\frac{1}{g^2}\,\sin^2(z)} \qquad ; \quad z_0 = 0 \quad , \quad z_1 = \frac{\pi}{2}$$

$$\begin{aligned} T_r^{(0)} &= \frac{\Gamma\left(r+\frac{1}{2}\right)^2}{\sqrt{\pi}\,\Gamma(r+1)} \\ &\sim \frac{(r-1)!}{\sqrt{\pi}}\left(1 - \frac{\frac{1}{4}}{(r-1)} + \frac{\frac{9}{32}}{(r-1)(r-2)} - \frac{\frac{75}{128}}{(r-1)(r-2)(r-3)} + \right. \end{aligned}$$

- low order coefficients about saddle z_1:

$$T^{(1)}(g^2) \sim i\,\sqrt{\pi}\left(1 - \frac{1}{4}\,g^2 + \frac{9}{32}\,g^4 - \frac{75}{128}\,g^6 + \dots\right)$$

- fluctuations about different saddles are explicitly related
- basic property of contour integrals
- could something like this work for path integrals ?

- periodic potential: $V(x) = \frac{1}{g^2}\,\sin^2(g\,x)$
- vacuum saddle point

$$c_n \sim n!\left(1 - \frac{5}{2}\cdot\frac{1}{n} - \frac{13}{8}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{2g^2}}\left(1 - \frac{5}{2}\cdot g^2 - \frac{13}{8}\cdot g^4 - \dots\right)$$

- double-well potential: $V(x) = x^2(1-gx)^2$
- vacuum saddle point

$$c_n \sim 3^n n!\left(1 - \frac{53}{6}\cdot\frac{1}{3}\cdot\frac{1}{n} - \frac{1277}{72}\cdot\frac{1}{3^2}\cdot\frac{1}{n(n-1)} - \dots\right)$$

- instanton/anti-instanton saddle point:

$$\mathrm{Im}\,E \sim \pi\,e^{-2\frac{1}{6g^2}}\left(1 - \frac{53}{6}\cdot g^2 - \frac{1277}{72}\cdot g^4 - \dots\right)$$

Connecting Perturbative and Non-perturbative Sectors

- in fact, there may be even more resurgent structure

- in certain non-trivial QM models, the perturbative series determines 1-instanton fluctuations, and beyond:

$$E(N,g^2) = E_{\rm pert}(N,g^2) \pm \frac{\left(\frac{2}{g^2}\right)^N}{\sqrt{2\pi g^2}\, N!} e^{-S/g^2}\, \mathcal{P}(N,g^2) \, + \ldots$$

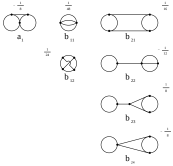
$$\mathcal{P}(N,g^2) = \frac{\partial E_{\rm pert}(N,g^2)}{\partial N} \exp \Bigg[S \int_0^{g^2} \frac{dg^2}{g^4} \Bigg(\frac{\partial E_{\rm pert}(N,g^2)}{\partial N} - 1 + \frac{\left(N+\frac{1}{2}\right)\, g^2}{S}$$

$\Rightarrow$ perturbation theory $E(N,g^2)$ encodes everything !

(GD, Ünsal, 1306.4405, 1401.5202)

Resurgence at Work

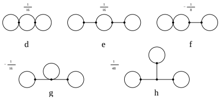
- fluctuations about $\mathcal{I}$ (or $\bar{\mathcal{I}}$) saddle are determined by those about the vacuum saddle, to all fluctuation orders



- "QFT computation": fluctuation about $\mathcal{I}$ for double-well (and periodic Mathieu potential):

3-loop (Escobar-Ruiz/Shuryak/Turbiner, arXiv:1501.03993)

$$e^{-\frac{S_0}{g^2}}\left[1-\frac{71}{72}\,g^2-0.607535\,g^4-\dots\right]$$



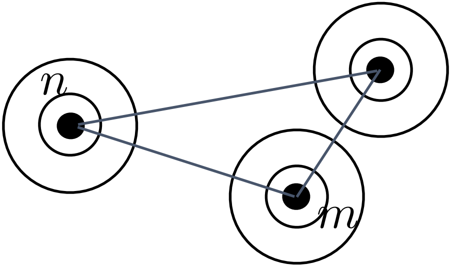
resurgence $\Rightarrow e^{-\frac{S_0}{g^2}}\left[1+\frac{1}{72}\,g^2\left(-102N^2-174N-71\right)\right.$

$$\left.+\frac{1}{10368}g^4\left(10404N^4+17496N^3-2112N^2-14172N-6299\right)+\dots\right]$$

- diagrammatically mysterious ...

Connecting Perturbative and Non-Perturbative Sectors

in certain non-trivial QM models, all orders of multi-instanton trans-series are encoded in perturbation theory of fluctuations about perturbative vacuum



- to see this, you need to include N: $energy = E(g^2, N)$
- e.g. application: SUSY QM

Resurgence and Large N: Double Trans-Series

- this talk: double trans-series

$$F = F(g^2) \quad \longrightarrow \quad F = F(g^2, N)$$

- instanton proliferation and strong-coupling instantons in the Mathieu spectrum ($\mathcal{N} = 2$ SUSY)

- large-N phase transitions: resurgence and complex saddles in Gross-Witten-Wadia matrix model

Physics Motivation

- large N expansion:

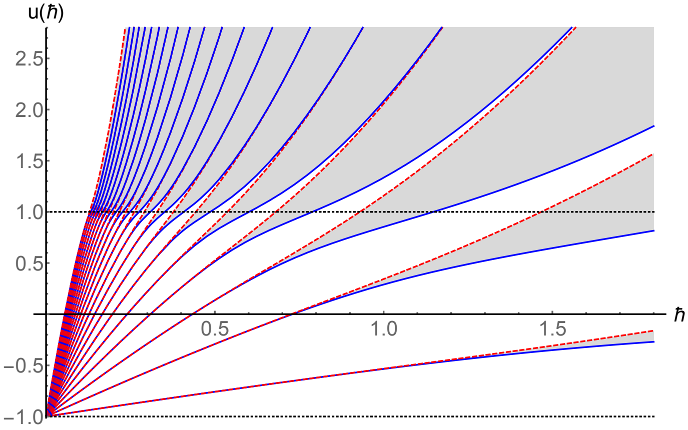
$$
\begin{aligned}
F(N,g^2) &= \sum_n g^{2n} p_n^{(0)}(N) \quad + \quad e^{-\frac{1}{g^2}} \sum_n g^{2n} p_n^{(1)}(N) + \dots \\
&= \sum_k \frac{1}{g^{2k}} c_k(N) \quad + \quad ? \\
&= \sum_{h=0}^{\infty} \frac{1}{N^{2h-2}} \, f_h(N\, g^2) \quad + \quad ?
\end{aligned}
$$

- how does a divergent trans-series at weak coupling turn into a convergent series at strong-coupling?
- non-perturbative corrections to $\frac{1}{N}$ expansion?
- separated by a phase transition: “instantons condense”

Mathieu Equation Spectrum: ($\hbar$ plays role of g^2)

- 3 different spectral regions: $-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2} + \cos(x)\,\psi = u\,\psi$

$$u = u(N, \hbar)$$



Resurgence of $\mathcal{N}=2$ SUSY SU(2) (Başar, GD, 1501.05671)

- moduli parameter: $u = \langle \mathrm{tr}\, \Phi^2 \rangle$
- electric: $u \gg 1$; magnetic: $u \sim 1$; dyonic: $u \sim -1$
- $a = \langle \text{scalar} \rangle$, $a_D = \langle \text{dual scalar} \rangle$, $a_D = \frac{\partial \mathcal{W}}{\partial a}$
- Nekrasov twisted superpotential $\mathcal{W}$
- Matone relation:

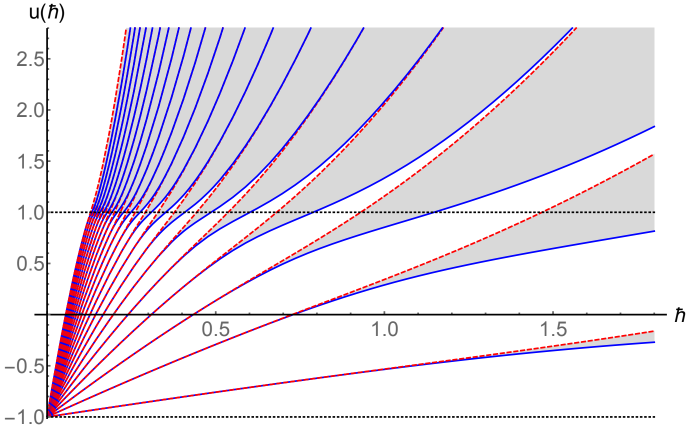
$$u(a, \hbar) = \frac{i\pi}{2} \Lambda \frac{\partial \mathcal{W}(a, \hbar, \Lambda)}{\partial \Lambda} - \frac{\hbar^2}{48}$$

- Mathieu equation:

$$-\frac{\hbar^2}{2} \frac{d^2 \psi}{dx^2} + \Lambda^2 \cos(x)\, \psi = u\, \psi \quad , \quad a \equiv \frac{N\hbar}{2}$$

- large a expansion acquires non-perturbative corrections

$$\mathcal{W} \longrightarrow \mathcal{W} + O(e^{-a})$$



electric

magnetic

dyonic

Mathieu Equation Spectrum

$$-\frac{\hbar^2}{2}\frac{d^2\psi}{dx^2}+\cos(x)\,\psi=u\,\psi$$

- small N: divergent, non-Borel-summable $\to$ trans-series

$$\begin{aligned}u(N,\hbar) \quad &\sim \quad -1+\hbar\left[N+\frac{1}{2}\right]-\frac{\hbar^2}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right]\\ &\qquad -\frac{\hbar^3}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right]-\dots\end{aligned}$$

- large N: convergent expansion: $\longrightarrow$?? trans-series ??

$$\begin{aligned}u(N,\hbar)\sim\frac{\hbar^2}{8}\Bigg(N^2&+\frac{1}{2(N^2-1)}\left(\frac{2}{\hbar}\right)^4+\frac{5N^2+7}{32(N^2-1)^3(N^2-4)}\left(\frac{2}{\hbar}\right)^8\\ &+\frac{9N^4+58N^2+29}{64(N^2-1)^5(N^2-4)(N^2-9)}\left(\frac{2}{\hbar}\right)^{12}+\dots\Bigg)\end{aligned}$$

Mathieu Equation Spectrum

- weak coupling:

$$u(N,\hbar) = -1 + \sum_{n=0}^{\infty} \hbar^n \, p_n(N)$$

- strong coupling:

$$u(N,\hbar) = -\frac{\hbar^2 N^2}{8} \sum_{n=0}^{\infty} \frac{1}{\hbar^{4n}} \, c_n(N)$$

- " 't Hooft coupling": $\lambda \equiv N \, \hbar$

- large N:

$$u(N,\lambda) = \sum_{n=0}^{\infty} \frac{1}{N^{2n}} \, F_n(\lambda)$$

- $F_n(\lambda)$ convergent at small λ, but divergent at large λ

$$u(N,\lambda) = \sum_{n=0}^{\infty} \frac{1}{N^{2n}} \, F_n(\lambda) + e^{-N\,S(\lambda)} \sum_{n=0}^{\infty} \frac{1}{N^{2n}} \, F_n^{(1)}(\lambda) + \dots$$

Mathieu Equation Spectrum

- exponentially narrow energy bands: $\lambda \ll 1$
- associated with real instantons

$$\Delta u^{\rm band}(N,\hbar) \quad \sim \quad \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} \exp\left[-\frac{8}{\hbar}\right]$$

- narrow energy gaps: $\lambda \gg 1$
- associated with complex instantons (Dykhne)

$$\Delta u^{\rm gap}(N,\hbar) \quad \sim \quad \frac{N\,\hbar^2}{2\pi}\left(\frac{e}{N\,\hbar}\right)^{2N} \sim \frac{N\,\hbar^2}{2\pi} e^{2N\ln(e/N\hbar)}$$

- universal non-perturbative splitting:

$$\Delta u(N,\hbar) \sim \frac{2}{\pi}\frac{\partial u}{\partial N}\,\exp\left[-\frac{2\pi}{\hbar}\,{\rm Im}\,a_D\right]$$

Beyond Large N: Multi-instantons at Strong Coupling

- $N\hbar \ll 1$, deep inside wells: resurgent trans-series

$$u^{(\pm)}(N,\hbar) \sim \sum_{n=0}^{\infty} c_n(N)\hbar^n \pm \frac{32}{\sqrt{\pi}\,N!}\left(\frac{32}{\hbar}\right)^{N-1/2} e^{-\frac{8}{\hbar}} \sum_{n=0}^{\infty} d_n(N)\hbar^n + \dots$$

- Borel poles at two-instanton location
- $d_n(N)$ encoded in $c_n(N)$
- $N\hbar \gg 1$, far above barrier: convergent series

$$u^{(\pm)}(N,\hbar) = \frac{\hbar^2 N^2}{8} \sum_{n=0}^{N-1} \frac{\alpha_n(N)}{\hbar^{4n}} \pm \frac{\hbar^2}{8} \frac{\left(\frac{2}{\hbar}\right)^{2N}}{(2^{N-1}(N-1)!)^2} \sum_{n=0}^{N-1} \frac{\beta_n(N)}{\hbar^{4n}} + \dots$$

(Basar, GD, Ünsal, 2015)

- perturbative fluctuations are finite polynomials
- $\beta_n(N)$ encoded in $\alpha_n(N)$

Reusrgence, Matrix Models, 2d Gauge Theory, Localization

$$F(N, g^2) = \sum_{h=0}^{\infty} \frac{1}{N^{2h-2}} F_h(N g^2)$$

- hermitean & unitary matrix models
- 2d Yang-Mills on sphere (Gross-Matytsin; Witten, Douglas-Kazakov)

$$Z(N, A) = \sum_R (\dim R)^2 \exp\left[-\frac{A}{2N} C_2(R)\right] \quad , \quad A_c = \pi^2$$

- certain protected quantities in especially symmetric QFTs can be reduced to matrix models $\Rightarrow$ resurgent asymptotics
- Chern-Simons on S^3: $Z(N, k) \longrightarrow$ matrix integral
- ABJM models on S^3: $Z(N, k) \longrightarrow$ matrix integral
- localizable 4d QFT: $Z(N, g^2) \longrightarrow$ matrix integral

Gross-Witten-Wadia Unitary Matrix Model

P. Buividovich, GD, S. Valgushev, 1512.09021

- integrate over $N \times N$ unitary matrices

$$Z = \int \mathcal{D}U \, \exp\left[\frac{N}{\lambda} \mathrm{Tr}\left(U + U^\dagger\right)\right]$$

- soluble for any finite N

- $N = \infty$: third order phase transition at $\lambda = 2$

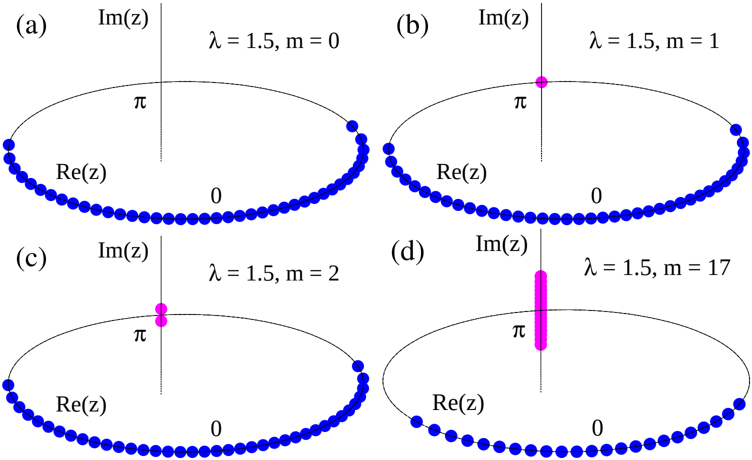
$$Z = \int_{-\pi}^{\pi} \prod_{i=1}^{N} dz_i \exp\left[-\frac{2N}{\lambda} \sum_i \cos(z_i) + \ln \prod_{i<j} \sin^2\left(\frac{z_i - z_j}{2}\right)\right]$$

- saddle point approach: $\partial S / \partial z_i = 0$

- which saddles (real or complex?) govern large N behavior? Resurgent structure?

- how to see phase transition at finite N?

Gross-Witten-Wadia Model: weak coupling: $\lambda < 2$



- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number

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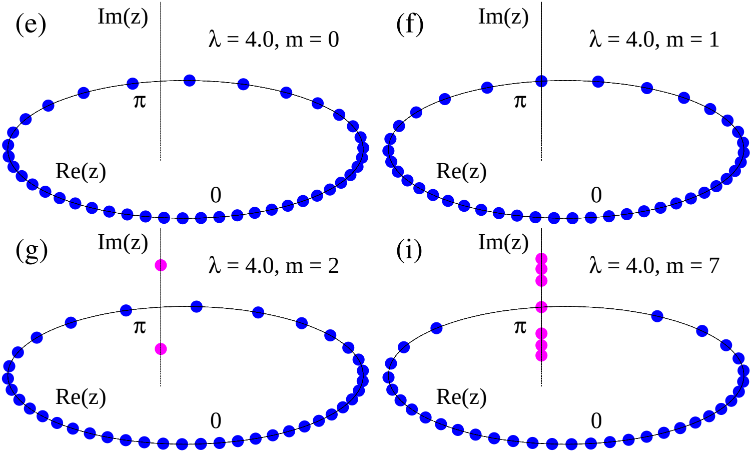
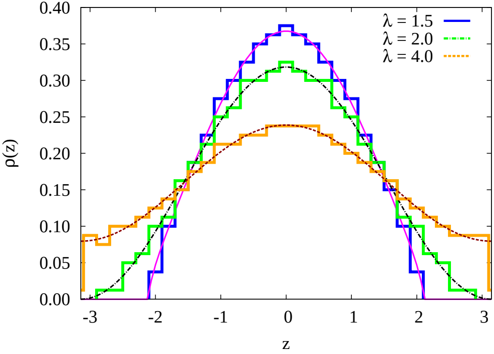


- "eigenvalue tunneling" of saddles into the complex plane
- number of complex eigenvalues: m = instanton number

Gross-Witten-Wadia Model: vacuum saddle (m=0)



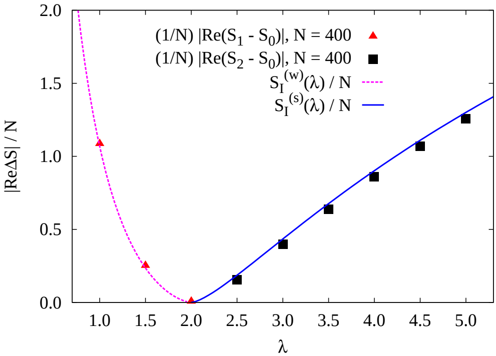
- $m = 0$: numerical eigenvalue dists.

$$\lambda < 2: \quad \rho^{(weak)}(z) = \frac{2}{\lambda\pi}\cos\left(\frac{z}{2}\right)\sqrt{\frac{\lambda}{2} - \sin^2\left(\frac{z}{2}\right)}$$

$$\lambda > 2: \quad \rho^{(strong)}(z) = \frac{1}{2\pi}\left(1 + \frac{2}{\lambda}\cos(z)\right)$$

Gross-Witten-Wadia Model: non-vacuum saddles

- weak coupling ($\lambda < 2$): $m = 1$ dominant
- strong coupling ($\lambda > 2$): $m = 2$ dominant



$$\lambda < 2 : \quad S_I^{(weak)} = 4/\lambda\sqrt{1-\lambda/2} - \operatorname{arccosh}\left((4-\lambda)/\lambda\right)$$

$$\lambda > 2 : \quad S_I^{(strong)} = 2\operatorname{arccosh}\left(\lambda/2\right) - 2\sqrt{1-4/\lambda^2}$$

- microscopic strong-coupling "instanton/saddle"

Resurgence in the Gross-Witten-Wadia Model (GD, A. Ahmed)

- weak coupling ($\lambda < 2$): Hessian has 1 negative mode

$\Rightarrow$ resurgent trans-series

$$-\ln Z(N,\lambda) \sim S_I^{(weak)}(\lambda) + \sum_n a_n \left(\frac{\lambda}{N}\right)^n + i\, e^{-4N/\lambda} \sum_n b_n \left(\frac{\lambda}{N}\right)^n + \dots$$

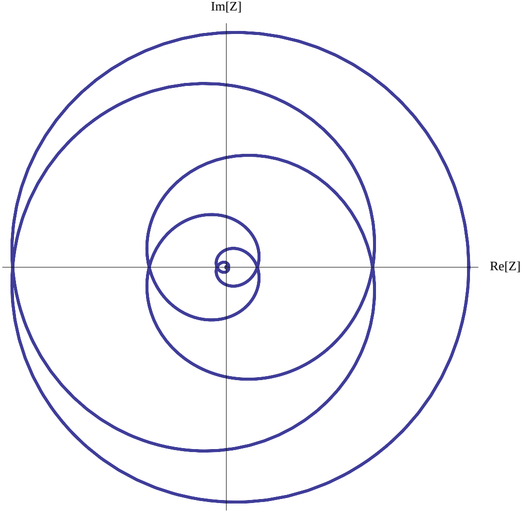
- explicit computation

$$a_n \sim \frac{\lambda}{4\pi} \frac{2^{N-1}}{N!} \frac{(n+N-1)!}{4^n} \left(1 + \delta \frac{4N/\lambda}{(n+N-1)} + \dots\right)$$

- leading behavior determines b_0
- subleading behavior determines b_1,   etc ...
- similarly for next exponential term,   etc ...
- resurgent trans-series at weak-coupling, large N

Analytic Continuation of GWW Model in λ

- intricate Stokes phenomenon as continue in phase of λ
- extremely sensitive to N and $|\lambda|$



Conclusions

- resurgent structure much richer for $F(g^2) \longrightarrow F(g^2, N)$
- reveals new non-perturbative effects at large N
- real and complex instantons; phase transitions; instanton proliferation; ...

- implications for sigma models and gauge theory ?